ON THE COHOMOLOGY OF CENTRAL FRATTINI EXTENSIONS

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ABSTRACT. In this paper we provide calculations for the mod p cohomology of certain p–groups, using topological methods. More precisely, we look at p-groups G defined as central extensions $1 \to V \to G \to W \to 1$ of elementary abelian groups such that $G/[G,G] \otimes \mathbb{F}_p = W$ and the defining k–invariants span the entire image of the Bockstein. We show that if $p > dim\ V - dim\ W + 1$, then the mod p cohomology of G can be explicitly computed as an algebra of the form $\mathcal{P} \otimes A$ where $\mathcal{P}$ is a polynomial ring on 2-dimensional generators and A is the cohomology of a compact manifold which in turn can be computed as the homology of a Koszul complex. As an application we provide a complete determination of the mod p cohomology of the universal central extension $1 \to H^2(W, \mathbb{F}_p) \to U \to W \to 1$ provided $p > \binom{n}{2} + 1$, where $n = dim\ W$.

1. INTRODUCTION

Obtaining a complete computation for the cohomology of a non–abelian p–group can be quite difficult. In fact very few general computations exist, especially in the case when p is an odd prime. In this paper we will consider certain central Frattini extensions of the form

$$1 \to (\mathbb{Z}/p)^v \to G \to (\mathbb{Z}/p)^w \to 1$$

where p is an odd prime. The condition that the extension be Frattini means that $G/[G,G] \otimes \mathbb{F}_p = (\mathbb{Z}/p)^w$.

Our main result will be that under appropriate conditions on the extension class, a *complete* calculation of the mod p cohomology of G can be obtained *provided* p is sufficiently large. Recall that a Frattini extension as above is uniquely determined by a subspace $K \subset H^2((\mathbb{Z}/p)^w, \mathbb{F}_p)$, where $H^*((\mathbb{Z}/p)^w, \mathbb{F}_p) \cong \Lambda(e_1, \ldots, e_w) \otimes \mathbb{F}_p[b_1, \ldots, b_w]$, with $|e_i| = 1$, and $b_i = \beta e_i$ (β is the Bockstein operator) for $i = 1, \ldots, w$. We use the notation above to state our main result.

Theorem 1.1. *Let G denote a finite p–group (p an odd prime) defined as a central extension $1 \to V \to G \to W \to 1$ of elementary abelian p–groups, where $v = dim\ V, w = dim\ W = dim\ H_1(G, \mathbb{Z}/p)$. Assume that the subspace $K \subset H^2(W, \mathbb{F}_p)$ defining the extension contains the entire image of the Bockstein, i.e., K has a basis of the form $\{b_1, \ldots, b_w, q_1, \ldots, q_{v-w}\}$, where $q_1, \ldots, q_{v-w} \in \Lambda(e_1, \ldots, e_w)$. Then the following hold*

- *Every element of order p in G is central.*
- *$H^*(G, \mathbb{F}_p) \cong \mathbb{F}_p[\zeta_1, \ldots, \zeta_v] \otimes H^*(M, \mathbb{F}_p)$ as $\mathbb{F}_p$-algebras, where M is the total space of a $(v - w)$-torus bundle over a w-torus and $\mathbb{F}_p[\zeta_1, \ldots, \zeta_v]$ is a polynomial algebra on generators ζ_i of degree two.*
- *If $p > v-w+1$, then the Lyndon-Hochschild-Serre spectral sequence for the extension above collapses at E_3 and*

$$H^*(G, \mathbb{F}_p)/(\zeta_1, \ldots, \zeta_v) \cong \mathrm{Tor}_{\mathbb{F}_p[c_1, \ldots, c_{v-w}]}(\Lambda(e_1, \ldots, e_w), \mathbb{F}_p)$$

where $|c_i| = 2$, $i = 1, \ldots, v - w$ and the Tor term is determined by the k–invariants, $c_i \mapsto q_i \in \Lambda(e_1, \ldots, e_w)$.

The Tor term can be explicitly understood as the homology of a Koszul complex, described as follows. Let $x_1, \ldots, x_{v-w}$ denote one–dimensional exterior classes, and denote by K the complex $\Lambda(e_1, \ldots, e_w) \otimes \Lambda(x_1, \ldots, x_{v-w})$, with differential determined by $\delta(x_i) = q_i$ for $i = 1, \ldots, v-w$. Then the Tor term above can simply be identified with the homology of K.

It would seem unlikely that a purely algebraic approach to group cohomology could have easily led to this result. Indeed one of the main drawbacks of general algebraic methods is the lack of geometric input required to determine ambiguities arising from specific differentials in the available spectral sequences (a basic computational device). In contrast, this type of problem has been considered by topologists in other situations pertaining to cohomology computations for bundles. In particular our approach provides a geometric alternative to the hypercohomology spectral sequence developed by Benson and Carlson (see [4]) in a very special situation. Our basic contribution is the observation that for certain group extensions G (such as those above) we can obtain a geometric model for the basis of $H^*(G, \mathbb{F}_p)$ as a free module over a polynomial subring of maximal rank.

This in turn allows us to introduce effective techniques from rational homotopy theory and from there outline conditions which imply the viability of a complete calculation (via spectral sequences). It would of course be interesting to recover the results here using purely algebraic methods, expressible perhaps in terms of minimal resolutions.

A motivation for this paper was the computation of the mod p cohomology of the *universal* central extension $U(n, p)$,

$$1 \to H^2((\mathbb{Z}/p)^n, \mathbb{Z}/p) \to U(n, p) \to (\mathbb{Z}/p)^n \to 1.$$

Applying our methods we obtain

Theorem 1.2. *Let $U(n, p)$ denote the universal central extension*

$$1 \to (\mathbb{Z}/p)^{\binom{n+1}{2}} \to U(n, p) \to (\mathbb{Z}/p)^n \to 1.$$

If $p > \binom{n}{2} + 1$, there is an exact sequence

$$0 \to (\zeta_1, \ldots, \zeta_{\binom{n+1}{2}}) \to H^*(U(n, p), \mathbb{F}_p) \to \mathrm{Tor}_{\mathbb{F}_p[c_{ij}]}(\Lambda(e_1, \ldots, e_n), \mathbb{F}_p) \to 0$$

where the Tor *term is determined by $c_{ij} \mapsto e_i e_j$ for $i < j$, $i, j = 1, \ldots, n$.*

Using an existing combinatorial computation, we can in fact make this explicit for most primes (see corollary 4.10). The groups $U(n, p)$ seem like basic objects in finite group cohomology. Further information on these examples would be very interesting, in particular it seems plausible to expect that the spectral sequence associated to the defining extension should collapse at E_3 for *all primes*.

The computations described here can be expressed as the collapse at E_2 of the Eilenberg-Moore spectral sequence associated to certain central extensions. However, in our situation this collapse is equivalent to a collapse at E_3 of the more familiar Lyndon–Hochschild–Serre spectral sequence (we will often abbreviate this as LHS); to simplify matters we have chosen to use this description.

This paper is organized as follows: in §2 we provide preliminary material required in our proofs, in §3 we describe how the cohomology calculations can be reduced to computing the cohomology of a compact manifold, in §4 we present the main result using the complete cohomology information available for large primes. In §5 we discuss formulas for the Bockstein.

Finally in §6 we raise a general question about the cohomology of central extensions of elementary abelian p–groups. Throughout this paper p will always denote an odd prime.

2. Preliminaries

In this section we will introduce the basic definitions and background needed to prove the results in this paper. We refer the reader to [3] and [8] for more details.

Definition 2.1. Let $1 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^l \to 1$ be a central extension. We shall say that it is *irreducible* if $H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^l$. Similarly we say that a central extension $1 \to (\mathbb{Z}/p)^r \to G \to (\mathbb{Z}/p)^l \to 1$ of elementary abelian p–groups (p a prime) is Frattini if $H_1(G, \mathbb{Z}/p) = (\mathbb{Z}/p)^l$.

Associated to any irreducible central extension as above, we have a subgroup

$$K \subset H^2(\Gamma/[\Gamma, \Gamma], \mathbb{Z})$$

which determines the extension unambiguously up to isomorphism. If

$$\kappa_1, \ldots, \kappa_r \in K$$

form a $\mathbb{Z}$–basis, then we say that they are a complete collection of k–invariants for the extension. A similar convention is made for central Frattini extensions of elementary abelian p–groups, using mod p coefficients.

Let $\pi : \mathbb{Z}^l \to (\mathbb{Z}/p^2)^l$ denote the natural mod p^2 quotient map. If $\mathcal{R} \subset H^*((\mathbb{Z}/p^2)^l, \mathbb{Z}/p)$ denotes the subring generated by one–dimensional classes, then π^* induces an isomorphism $\mathcal{R} \to H^*(\mathbb{Z}^l, \mathbb{Z}/p)$.

Definition 2.2. Let $K \subset H^2(\Gamma/[\Gamma, \Gamma], \mathbb{Z})$, Γ as above. We define $K_p \subset H^2(\Gamma/[\Gamma, \Gamma] \otimes \mathbb{Z}/p^2, \mathbb{Z}/p)$ as $(\pi^*)^{-1}([K])$, where $[K]$ is the mod p reduction of K.

This defines a central extension

$$1 \to (\mathbb{Z}/p)^r \to G(p) \to (\mathbb{Z}/p^2)^l \to 1$$

which also fits into an extension of the form

$$1 \to \Gamma(p) \to \Gamma \to G(p) \to 1$$

derived from the natural projection.

Central extensions can also be studied geometrically, in particular the group Γ has a classifying space which is the homotopy fiber of a map

$$\psi : (\mathbb{S}^1)^l \to (\mathbb{C}P^\infty)^r.$$

Let $\rho : (\mathbb{S}^1)^t \to (\mathbb{S}^1)^t$ be the p–th power map in each coordinate; it will induce a map $B\rho : (\mathbb{C}P^\infty)^t \to (\mathbb{C}P^\infty)^t$. We can describe the classifying space of $\Gamma(p)$ as the homotopy fiber of

$$B\rho \cdot \psi \cdot \rho : (\mathbb{S}^1)^l \to (\mathbb{C}P^\infty)^r.$$

Note that this map is trivial in mod p cohomology, hence the resulting k–invariants are trivial mod p.

We examine the p–group $G(p)$, but first we recall a definition.

Definition 2.3. A finite group G is said to satisfy the pC condition if every element of order p in G is central.

Proposition 2.4. $G(p)$ *is a finite p–group satisfying the pC condition which can be expressed as a Frattini extension*

$$1 \to (\mathbb{Z}/p)^{r+l} \to G(p) \to (\mathbb{Z}/p)^l \to 1.$$

Proof. By construction the k–invariants defining $G(p)$ are decomposable (as sums of products of one–dimensional classes), hence they are elements in $H^2((\mathbb{Z}/p^2)^l, \mathbb{Z}/p)$ which restrict to zero on the subgroup $(\mathbb{Z}/p)^l \subset (\mathbb{Z}/p^2)^l$. Thus the extension restricted to this subgroup defines an elementary abelian subgroup of rank equal to $r+l$ which is the kernel of the natural projection $G(p) \to (\mathbb{Z}/p)^l$. This of course has maximal rank in $G(p)$ and is central, hence the proposition follows. $\square$

We know by [7] that the groups $G(p)$ have Cohen–Macaulay mod p cohomology, i.e. $H^*(G(p), \mathbb{F}_p)$ is free and finitely generated over a polynomial subalgebra on $r+l$ generators. The following lemma (a version of which appears in [2]) explains how to locate a regular sequence; we include a proof for the sake of completeness.

Lemma 2.5. *Let G denote a finite p-group satisfying the pC condition with $E \subseteq G$ the elementary abelian subgroup of maximal rank n. Let $\mathcal{P} \subset H^*(G, \mathbb{F}_p)$ be a polynomial subalgebra such that $H^*(E, \mathbb{F}_p)$ is a finitely generated module over $res_E^G(\mathcal{P})$. Then $H^*(G, \mathbb{F}_p)$ is a free and finitely generated module over $\mathcal{P}$.*

Proof. Let $\mathcal{P} = \mathbb{F}_p[\zeta_1, \ldots, \zeta_n]$, we know that $H^*(G, \mathbb{F}_p)$ is Cohen-Macaulay and by a standard result in commutative algebra it follows that under that condition the cohomology will be a free module over any polynomial subring over which it is finitely generated (see [12]). Hence we only need to prove that $H^*(G, \mathbb{F}_p)$ is a finitely generated $\mathcal{P}$-module. To prove this we will use the more geometric language of cohomological varieties (see [8] for background).

Let $V_G(\zeta_i)$ denote the homogeneous hypersurface in V_G (the maximal ideal spectrum for $H^*(G, \mathbb{F}_p)$) defined by ζ_i. Then $H^*(G, \mathbb{F}_p)$ will be finitely generated over $\mathcal{P}$ if and only if $V_G(\zeta_1) \cap \cdots \cap V_G(\zeta_n) = \{0\}$. If we represent the class ζ_i by an epimorphism $\Omega^{n_i}(\mathbb{F}_p) \to \mathbb{F}_p$ with kernel L_{ζ_i}, then we know that $V_G(\zeta_i) = V_G(L_{\zeta_i})$, the variety associated

to the annihilator of $Ext^*_{\mathbb{F}_p G}(L_{\zeta_i}, L_{\zeta_i})$. Moreover using basic properties of these varieties, we have that $V_G(\zeta_1) \cap \cdots \cap V_G(\zeta_n) = V_G(L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_i})$. Now the cohomological variety of a module will be 0 if and only if the module is projective, hence what we need to prove is that the module $L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_n}$ is projective. However by Chouinard's Theorem (see [8]) we know that it is enough to check this by restricting to maximal elementary abelian subgroups; in this case E is the only such group and projectivity follows from our hypothesis, as $\mathrm{res}^G_E(\mathcal{P}) \subseteq H^*(E, \mathbb{F}_p)$ is a polynomial subalgebra over which it is finitely generated (note that by Quillen's detection theorem, the kernel of res^G_E is nilpotent, hence $\mathcal{P}$ embeds in $H^*(E)$ under this map). Hence we conclude that $\zeta_1, \ldots, \zeta_n$ form a homogeneous system of parameters and so $H^*(G, \mathbb{F}_p)$ is free and finitely generated as a module over $\mathcal{P}$.   □

3. Cohomology Calculations

The goal of this section will be to analyze and in some instances compute the cohomology ring of the p–groups $G(p)$. We begin by recording the cohomology of $\Gamma(p)$, which follows directly from the mod p triviality of its defining k–invariants.

Proposition 3.1. *Given $\Gamma(p)$ as above, its mod p cohomology is an exterior algebra on $r + l$ one–dimensional generators.*

Proof. Indeed $\Gamma(p)$ can be expressed as a central extension

$$1 \to \mathbb{Z}^r \to \Gamma(p) \to \mathbb{Z}^l \to 1$$

which by construction yields a mod p LHS spectral sequence which collapses at E_2.   □

Definition 3.2. Given a pC group G, let $\Omega_1(G)$ be the maximal elementary abelian subgroup of G. We will let $H^*(\Omega_1(G), \mathbb{F}_p)_{red}$ be the quotient algebra of $H^*(\Omega_1(G), \mathbb{F}_p)$ modulo the ideal of nilpotent elements. This is a graded polynomial algebra on degree 2 generators.

Definition 3.3. A pC group G is said to have the Ω_1-extension property if the restriction $H^2(G; \mathbb{F}_p) \to H^2(\Omega_1(G); \mathbb{F}_p)_{red}$ is onto.

The Ω_1-extension property was studied in [14] and [5]. We will need the following theorem of T. Weigel which was proven in [14] using Hopf algebra techniques. It represents a strengthening of the Cohen-Macaulay property we previously explained:

Theorem 3.4. *If G is a pC group with the Ω_1-extension property, then we have an isomorphism of $\mathbb{F}_p$-algebras*

$$H^*(G; \mathbb{F}_p) \cong \mathbb{F}_p[\zeta_1, \ldots, \zeta_n] \otimes C^*$$

for some finite dimensional algebra C^*. Here $n = \dim \Omega_1(G)$ and the ζ_i are degree 2 elements which restrict to a generating set of $H^2(\Omega_1(G), \mathbb{F}_p)_{red}$.

Notice in theorem 3.4, that if $\eta_1, \ldots, \eta_n$ is another regular sequence of degree 2 elements which restricts to the generators of $H^2(\Omega_1(G), \mathbb{F}_p)_{red}$, we have that

$$(\eta_1, \ldots, \eta_n) = (\zeta_1 + c_1, \ldots, \zeta_n + c_n)$$

for some $c_i \in C^2$, $1 \leq i \leq n$.

However by the universal property of the tensor product of algebras, we can define a graded algebra endomorphism Ψ of $H^*(G, \mathbb{F}_p)$ by $\Psi(\zeta_i) = \zeta_i + c_i$ for all $1 \leq i \leq n$ and $\Psi|_{C^*} = Id_{C^*}$. Since we can construct an inverse for Ψ analogously, Ψ is a graded algebra automorphism of $H^*(G, \mathbb{F}_p)$ which takes the ideal $(\zeta_1, \ldots, \zeta_n)$ to the ideal $(\eta_1, \ldots, \eta_n)$. Thus we conclude that we have an isomorphism of graded algebras

$$H^*(G, \mathbb{F}_p)/(\eta_1, \ldots, \eta_n) \cong C^*$$

for any such regular sequence. Furthermore, $H^*(G, \mathbb{F}_p) \cong \mathbb{F}_p[\eta_1, \ldots, \eta_n] \otimes C^*$ as algebras.

Our next step is to locate an explicit regular sequence in $G(p)$.

Proposition 3.5. *There is a regular sequence of maximal length in $H^*(G(p), \mathbb{F}_p)$ given by elements $\zeta_1, \ldots, \zeta_{r+l} \in H^2(G(p), \mathbb{F}_p)$ which restrict to generators of $H^2(\Omega_1(G(p)), \mathbb{F}_p)_{red}$. Thus $G(p)$ has the Ω_1-extension property.*

Proof. We consider the Lyndon–Hochschild–Serre spectral sequence with mod p coefficients for the group extension

$$1 \to (\mathbb{Z}/p)^{r+l} \to G(p) \to (\mathbb{Z}/p)^l \to 1.$$

Recall that $H^*((\mathbb{Z}/p)^l, \mathbb{F}_p) \cong \Lambda(e_1, \ldots, e_l) \otimes \mathbb{F}_p[b_1, \ldots, b_l]$, where, if β denotes the usual Bockstein operator, $\beta(e_i) = b_i$ for $i = 1, \ldots, l$. If $\gamma : (\mathbb{Z}/p^2)^l \to (\mathbb{Z}/p)^l$ is the natural projection, it induces an isomorphism between the subrings of the mod p cohomologies generated by 1-dimensional classes. The k–invariants defining the extension $G(p)$ are precisely $(\gamma^*)^{-1}(K_p) \cup \{b_1, \ldots, b_l\}$. As a consequence of this the differential d_3 is zero on the two–dimensional polynomial generators in the fiber (indeed the ideal generated by the transgressions of the one–dimensional classes includes the entire ideal generated by the $b_1, \ldots, b_l$, hence the Bocksteins of these transgressions are in the ideal already).

Now choose $\zeta_1, \ldots, \zeta_{r+l}$ to be classes in $H^2(G(p), \mathbb{F}_p)$ restricting to the two–dimensional permanent cocycle polynomial classes in the edge of the spectral sequence. These are our desired elements. □

We now assemble the cohomology information above to understand the cohomology of the finite p–groups $G(p)$.

Theorem 3.6. *In the mod p LHS spectral sequence associated to*

$$1 \to \Gamma(p) \to \Gamma \to G(p) \to 1$$

the action of $G(p)$ is homologically trivial and the one–dimensional generators in the cohomology of $\Gamma(p)$ transgress to a regular sequence $\{\zeta_i\}_{i=1}^{r+l}$ in $H^2(G(p), \mathbb{F}_p)$, $E_3 = E_\infty$, and in particular if I is the ideal generated by these transgressions, then
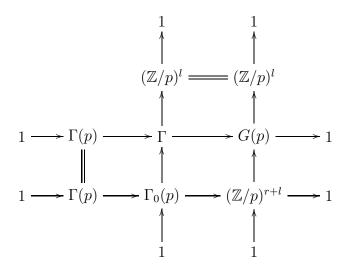
$$H^*(G(p), \mathbb{F}_p)/I \cong H^*(\Gamma, \mathbb{F}_p).$$

Furthermore we have an isomorphism of $\mathbb{F}_p$-algebras,

$$H^*(G(p), \mathbb{F}_p) \cong P^* \otimes H^*(\Gamma, \mathbb{F}_p).$$

where P^* is a polynomial algebra on $(r + l)$ degree 2 generators.

Proof. Let $\Gamma_0(p)$ denote the p–Frattini subgroup of Γ, i.e., the kernel of the natural projection $\Gamma \to (\mathbb{Z}/p)^l$. This group fits into a commutative diagram of extensions:

Note that $H = (\mathbb{Z}/p)^{r+l} \subset G(p)$ is $\Omega_1(G(p))$. As before, the k–invariants defining $\Gamma_0(p)$ are trivial mod p, hence $H^*(\Gamma_0(p), \mathbb{F}_p)$ is also an exterior algebra on $r + l$ one–dimensional

generators, and the map $\phi : \Gamma_0(p) \to H$ induces a surjection in mod p cohomology. In particular, if we let

$$H^*(H, \mathbb{F}_p) = \Lambda(e_1, \ldots, e_{r+l}) \otimes \mathbb{F}_p[\beta e_1, \ldots, \beta e_{r+l}], \ \ H^*(\Gamma_0(p), \mathbb{F}_p) = \Lambda(y_1, \ldots, y_{r+l})$$

then we can assume that $\phi^*(e_i) = y_i$, $\phi^*(\beta e_i) = \beta y_i$. On the other hand we can choose $q_i \in \Lambda(e_1, \ldots, e_{r+l})$ such that $\phi^*(q_i) = \beta y_i$. Hence we obtain the following basis for the kernel of ϕ^* in dimension equal to two: $\{b_1 - q_1, \ldots, b_{r+l} - q_{r+l}\}$.

Now consider the mod p LHS spectral sequence for the bottom row. Since, $H^1(H, \mathbb{F}_p) \cong H^1(\Gamma_0(p), \mathbb{F}_p)$, the five term exact sequence associated to this spectral sequence shows that

$$H^1(\Gamma(p), \mathbb{F}_p)^H \stackrel{d_2}{\cong} ker(\phi^* : H^2(H, \mathbb{F}_p) \to H^2(\Gamma_0(p), \mathbb{F}_p)).$$

By comparing dimensions, we see that H must act trivially on $H^1(\Gamma(p), \mathbb{F}_p)$ and hence on $H^*(\Gamma(p), \mathbb{F}_p)$. Furthermore, d_2 embeds $H^1(\Gamma(p), \mathbb{F}_p)$ naturally as a subspace of $H^2(H, \mathbb{F}_p)$. Since H is central in $G(p)$, we have $G(p)$ acts trivially on $H^*(H, \mathbb{F}_p)$ and we conclude that $G(p)$ acts trivially on $H^1(\Gamma(p), \mathbb{F}_p)$ by the natural d_2-embedding above, and hence $G(p)$ acts trivially on all of $H^*(\Gamma(p), \mathbb{F}_p)$.

Let I' denote the ideal generated by the d_2-transgressions. Then evidently I' is generated by the basis of $ker(\phi^*)$ above, which one can easily verify to be a regular sequence in the cohomology of H.

Next we consider the spectral sequence for Γ given by the middle row of our diagram. We have seen that $G(p)$ acts trivially on $H^*(\Gamma(p), \mathbb{F}_p)$. Comparing spectral sequences, we see that I is generated by trangressions ζ_i, $i = 1, \ldots, r + l$ which restrict to a regular sequence of maximal length in $H^*(H, \mathbb{F}_p)$.

By theorem 3.4 and the comments immediately following it, $H^*(G(p)) \cong P^* \otimes C^*$ where P^* is a polynomial algebra on $\zeta_1, \ldots, \zeta_{r+l}$. From this, it is easy to compute that $E_3 = C^*$ and hence $E_3 = E_\infty$. Since E_∞ is concentrated on a single horizontal row, there are no problems in lifting the ring structure to that of $H^*(\Gamma, \mathbb{F}_p)$. Hence we conclude that

$$H^*(\Gamma, \mathbb{F}_p) \cong H^*(G(p), \mathbb{F}_p)/(\zeta_1, \ldots, \zeta_{r+l}) \cong C^*.$$

This completes the proof. $\qquad\square$

Corollary 3.7. *Let $q(t)$ denote the Poincaré series for the mod p cohomology of Γ and $p(t)$ the one for the mod p cohomology of $G(p)$. Then we have $p(t) = \frac{q(t)}{(1-t^2)^{r+l}}$.*

In the next section we examine the cohomology of Γ using topological methods.

4. Cohomology of Γ for Large Primes

As we saw in the previous section, the computation of the $\mathbb{F}_p$-cohomology ring of $G(p)$ has been reduced to understanding that of Γ. It turns out that for sufficiently large primes we have a very precise algebraic description of the cohomology groups, obtained using bundle theory. As described in [11], an explicit model for the classifying space $B\Gamma$ can be easily given; it will be a compact manifold described as a bundle over a generalized torus with fibre another generalized torus. This in turn can be described (up to homotopy) by a map from a generalized torus to a product of infinite complex projective spaces. The basic calculational result is summarized in

Theorem 4.1. *Let Γ be defined as a central extension of the form $1 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^l \to 1$, defined by a map $\psi : (\mathbb{S}^1)^l \to (\mathbb{C}P^\infty)^r$. Let $p > r + 1$ be any prime number, then*

$$H^*(\Gamma, \mathbb{F}_p) \cong \mathrm{Tor}_{\mathbb{F}_p[c_1,\ldots,c_r]}(\Lambda(e_1, \ldots, e_l), \mathbb{F}_p),$$

where the Tor *term is determined by $\psi^*(c_1), \ldots, \psi^*(c_r) \in \Lambda(e_1, \ldots, e_l)$, and $\{e_1, \ldots, e_l\}$ are one–dimensional generators for the exterior algebra $H^*((\mathbb{S}^1)^l, \mathbb{F}_p)$ and $\{c_1, \ldots, c_r\}$ are two–dimensional generators for the polynomial algebra $H^*((\mathbb{C}P^\infty)^r, \mathbb{F}_p)$.*

Proof. By a result due to Lambe and Priddy (see [11]), the LHS spectral sequence associated to the defining extension above collapses at E_3 if coefficients are taken in $\mathbb{Z}_{(p)}$ for p sufficiently large. The improved lower bound $p > r + 1$ was obtained in [6]. This implies the collapse of the mod p spectral sequence, the statement is readily derived from the algebraic interpretation of this E_3–term. $\qquad\qquad\square$

Corollary 4.2. *If p is sufficiently large, then*

$$H^*(\Gamma, \mathbb{F}_p) \cong \mathrm{Tor}_{\mathbb{Z}[c_1,\ldots,c_r]}(\Lambda_{\mathbb{Z}}(e_1, \ldots, e_l), \mathbb{Z}) \otimes \mathbb{F}_p.$$

Proof. Indeed, the integral Tor term can only have p–torsion for finitely many primes p, the result follows from the mod p reduction sequence for Tor. $\qquad\qquad\square$

Remark 4.3. In fact [11] and [6] prove that the $\mathbb{Z}_{(p)}$–cohomology of Γ for p sufficiently large is isomorphic to the cohomology of the Koszul complex associated to a certain Lie algebra; this is precisely the E_3–term of the LHS spectral sequence.

We can now state the main result in this paper, which puts together the different facts we have proved.

Theorem 4.4. *Let G denote a finite p–group (p an odd prime) defined as a central extension $1 \to V \to G \to W \to 1$ of elementary abelian p–groups, where $v = dim\ V, w =$*

dim $W = dim\ H_1(G, \mathbb{Z}/p)$ *and* $H^*(W, \mathbb{F}_p) \cong \Lambda(e_1, \ldots, e_w) \otimes \mathbb{F}_p[b_1, \ldots, b_w]$. *Assume that the subspace* $K \subset H^2(W, \mathbb{F}_p)$ *defining the extension contains the entire image of the Bockstein, i.e.,* K *has a basis of the form* $\{b_1, \ldots, b_w, q_1, \ldots, q_{v-w}\}$, *where* $q_1, \ldots, q_{v-w} \in \Lambda(e_1, \ldots, e_w)$. *Then the following hold*

- *Every element of order* p *in* G *is central.*
- *We have an isomorphism of* $\mathbb{F}_p$-*algebras:*

$$H^*(G, \mathbb{F}_p) \cong \mathbb{F}_p[\zeta_1, \ldots, \zeta_v] \otimes H^*(M, \mathbb{F}_p)$$

  where M *is the total space of a* $(v - w)$-*torus bundle over a* w-*torus and the degree of the* ζ_i *is two.*
- *If* $p > v - w + 1$, *then the LHS spectral sequence for the extension above collapses at* E_3 *and*

$$H^*(G, \mathbb{F}_p)/(\zeta_1, \ldots, \zeta_v) \cong \mathrm{Tor}_{\mathbb{F}_p[c_1, \ldots, c_{v-w}]}(\Lambda(e_1, \ldots, e_w), \mathbb{F}_p)$$

  where $|c_i| = 2$, $i = 1, \ldots, v - w$ *and the* Tor *term is determined by the* k–*invariants,* $c_i \mapsto q_i \in \Lambda(e_1, \ldots, e_w)$.

Remark 4.5. Note that it has been shown (see [1]) that the cohomology of a p–group satisfying the pC condition cannot be detected on proper subgroups, hence this is an intrinsic calculation.

Remark 4.6. The Tor algebra which appears above can be identified with the mod p cohomology of a compact manifold (as before). Its fundamental group Γ maps onto G inducing a surjection in mod p cohomology.

We will now exhibit an interesting class of groups to which this theorem can be applied. In fact the computation of the cohomology of these groups was a motivation for this work.

Example 4.7. An easy dimension–count shows that

$$\dim\ H^2((\mathbb{Z}/p)^n, \mathbb{Z}/p) = \binom{n+1}{2}.$$

Hence we may construct a Frattini central extension of the form

$$1 \to (\mathbb{Z}/p)^{\binom{n+1}{2}} \to U(n, p) \to (\mathbb{Z}/p)^n \to 1$$

which has the following universal property: given any other central Frattini extension of the form

$$1 \to V \to U \to (\mathbb{Z}/p)^n \to 1$$

there exists a central elementary abelian subgroup $E \subset U(n,p)$ such that $U(n,p)/E \cong U$. Given this basic property of $U(n,p)$, a natural problem to consider is the computation of its mod p cohomology. Our results imply a complete answer for p sufficiently large.

Definition 4.8. Define $H(n)$ to be the universal central extension

$$1 \to H^2(\mathbb{Z}^n, \mathbb{Z}) \to H(n) \to \mathbb{Z}^n \to 1.$$

$H(n)$ is sometimes called the free two-step nilpotent group on n generators.

Theorem 4.9. *If $U(n,p)$ denotes the universal central extension*

$$1 \to (\mathbb{Z}/p)^{\binom{n+1}{2}} \to U(n,p) \to (\mathbb{Z}/p)^n \to 1,$$

then there is an isomorphism of graded algebras

$$H^*(U(n,p), \mathbb{F}_p) \cong \mathbb{F}_p[\zeta_1, \ldots, \zeta_{\binom{n+1}{2}}] \otimes H^*(H(n); \mathbb{F}_p).$$

If $p > \binom{n}{2} + 1$, then the cohomology groups above are determined by the isomorphism

$$H^*(H(n), \mathbb{F}_p) \cong \mathrm{Tor}_{\mathbb{F}_p[c_{ij}]}(\Lambda(e_1, \ldots, e_n), \mathbb{F}_p)$$

where the module structure is specified by $c_{ij} \mapsto e_i e_j$ for $i < j$, $i,j = 1, \ldots, n$.

It so happens that the corresponding rational Tor terms have been computed using representation theory (see [10], [13]). From this we obtain the following

Corollary 4.10. *Let $U(n,p)$ be the universal central extension. Then, for a fixed integer $n > 0$ and for all but a finite number of primes p, the Poincaré series for $H^*(U(n,p), \mathbb{F}_p)$ is given by $r(t) = \frac{q(t)}{(1-t^2)^m}$ where, if $q(t) = 1 + a_1 t + \cdots + a_m t^m$, $m = \binom{n+1}{2}$, then*

$$a_i = \sum_{f+g=i} \sum_{Y_\lambda} \prod_{(s,t) \in Y_\lambda} \frac{n + t - s}{h(s,t)}$$

where Y_λ ranges over all symmetric, $f + 2g$-box, f-hook Young diagrams, and $h(s,t)$ denotes the hooklength of the box (s,t). In particular we have

$$a_1 = n, \quad a_2 = \frac{n(n+1)(n-1)}{3}, \quad a_3 = \frac{n(n^2 - 1)(3n - 4)(n + 3)}{60}.$$

Notice how we have obtained a complete calculation for almost every prime. This type of result would be extremely difficult to observe using traditional methods in group cohomology, or even computer–assisted calculations. We have in fact outlined an *effective* method for describing a basis for the cohomology of an important class of p–groups as modules over a polynomial subalgebra on 2–dimensional classes.

5. Bockstein on $H^*(U(n,p), \mathbb{F}_p)$

Let $\mathfrak{h}(n)$ be the $\mathbb{F}_p$-Lie algebra with basis elements $\{e_k, e_{i,j}\}$ where the indices i, j, k range over the set $\{1, \ldots, n\}$ and we insist that $i < j$.

The Lie bracket is given by

$$[e_i, e_j] = e_{i,j} \text{ and } e_{i,j} \text{ is central}$$

for $1 \leq i < j \leq n$. Notice that $\mathfrak{h}(n)$ has dimension $\binom{n+1}{2}$. Also notice that the Tor term in Theorem 4.9 is exactly the Lie algebra cohomology of $\mathfrak{h}(n)$.

As in [5] and work of T. Weigel, to every $\mathbb{F}_p$-Lie algebra $\mathfrak{L}$ there corresponds a p-group of exponent p^2 satisfying the pC condition.

To construct this group, one takes a free $\mathbb{Z}/p^2$-module K of rank equal to the dimension of $\mathfrak{L}$. Then one has a canonical surjection $\pi : K \to \mathfrak{L}$ and injection $i : \mathfrak{L} \to K$ such that $i \circ \pi$ is multiplication by p on K. One defines the group structure on K by

$$x \cdot y = x + y + i([\pi(x), \pi(y)])$$

for all $x, y \in K$.

Let $G(\mathfrak{h}(n))$ be the p-group associated in this manner to $\mathfrak{h}(n)$. Then $G(\mathfrak{h}(n))$ fits into a central Frattini extension

$$1 \to V \to G(\mathfrak{h}(n)) \xrightarrow{\pi} W \to 1$$

where V and W are elementary abelian p-groups of rank $\binom{n+1}{2}$.

Identifying W with $\mathfrak{h}(n)$ as in [5], we may look at the subspace $S \subset W$ spanned by $\{e_k\}_{1 \leq k \leq n}$. Using this description, one can verify easily that $\pi^{-1}(S) \cong U(n,p)$. Thus $U(n,p)$ is a subgroup of $G(\mathfrak{h}(n))$ which contains $\Omega_1(G(\mathfrak{h}(n)))$.

From the results in [5], it follows that we have an isomorphism of algebras

$$H^*(G(\mathfrak{h}(n)), \mathbb{F}_p) \cong \mathbb{F}_p[\zeta_k, \zeta_{i,j}] \otimes \wedge^*(x_k, x_{i,j})$$

where the indices i, j, k range over the set $\{1, \ldots, n\}$ and we always have $i < j$.

Furthermore for $p > 3$, the Bockstein is given by the following formulas:

$$\beta(x_i) = 0 \text{ and } \beta(x_{i,j}) = -x_i x_j$$
$$\beta(\zeta_i) = 0 \text{ and } \beta(\zeta_{i,j}) = \zeta_i x_j - \zeta_j x_i.$$

It was also shown that the ζ-elements restrict to a regular sequence in the $\mathbb{F}_p$-cohomology of $\Omega_1(G(\mathfrak{h}(n))$.

Now using that $U(n,p)$ is a subgroup of $G(\mathfrak{h}(n))$ and that the Ω_1 subgroups of these two groups coincide, by our previous results we have

$$H^*(U(n,p), \mathbb{F}_p) \cong \mathbb{F}_p[\zeta_k, \zeta_{i,j}] \otimes H^*(H(n), \mathbb{F}_p)$$

where the polynomial algebra here is the image of the corresponding one for $G(\mathfrak{h}(n))$ and we have abused notation and identified the ζ elements with their restrictions. Notice also that on the $H^1(-, \mathbb{F}_p)$-level, the restriction map from $H^*(G(\mathfrak{h}(n)), \mathbb{F}_p)$ to $H^*(U(n,p), \mathbb{F}_p)$ takes $x_{i,j}$ to zero and $\{x_k\}_{k=1}^n$ to a basis of $H^1(U(n,p), \mathbb{F}_p)$. Furthermore, the formulas for the Bockstein above restrict with the obvious identifications. Thus we obtain:

Proposition 5.1. *For $p > 3$,*

$$H^*(U(n,p), \mathbb{F}_p) \cong \mathbb{F}_p[s_k, s_{i,j}] \otimes H^*(H(n), \mathbb{F}_p)$$

as algebras where as usual the indices i, j, k range over the set $\{1, \ldots, n\}$ and we always have $i < j$. Furthermore there is a basis $\{x_1, \ldots, x_n\}$ of $H^1(U(n,p), \mathbb{F}_p)$ such that

$$\beta(s_k) = 0 \ \text{and} \ \beta(s_{i,j}) = s_i x_j - s_j x_i$$
$$\beta(x_k) = 0 \ \text{and} \ x_i x_j = 0$$

for all $1 \leq i < j \leq n$, $1 \leq k \leq n$.

Remark 5.2. For large primes p, the Bockstein on $H^*(H(n), \mathbb{F}_p)$ vanishes since the integral homology of $H(n)$ is finitely generated. However we still cannot conclude that proposition 5.1 gives us the complete structure of the Bockstein on $H^*(U(n,p), \mathbb{F}_p)$ since the subalgebra of $H^*(U(n,p), \mathbb{F}_p)$ corresponding to $H^*(H(n), \mathbb{F}_p)$ under the isomorphism above is not in general closed under the higher Bocksteins.

6. Final Remarks and a Problem

The results in this paper can be thought of as special cases of a more general collapse theorem for spectral sequences. In fact, given a central extension

$$1 \to V \to G \to W \to 1$$

where V and W are both elementary abelian p–groups, there is an Eilenberg–Moore spectral sequence (see [9]) with E_2 term equal to $\text{Tor}_{H^*(K(V,2), \mathbb{F}_p)}(H^*(W, \mathbb{F}_p), \mathbb{F}_p)$ and converging to $H^*(G, \mathbb{F}_p)$. Historically these spectral sequences have been most useful when they collapse at E_2. To the best of our knowledge this occurs for every group extension of this type whose cohomology has been computed. This leads us to raise the following somewhat difficult

Question 6.1. Let

$$1 \to V \to G \to W \to 1$$

denote a central extension where both V and W are elementary abelian p–groups. Can the mod p Eilenberg–Moore spectral sequence fail to collapse at E_2? If so, give reasonable conditions on the k–invariants (as above) which imply a collapse.

REFERENCES

[1] Adem, A. and Karagueuzian, D. *Essential Cohomology of Finite Groups*, Comm. Math. Helv. **72** (1997), pp. 101–109.

[2] Adem, A., Karagueuzian, D. and Minac, J. *On the Cohomology of Galois Groups Determined by Witt Rings*, Advances in Mathematics **148** (1999), pp. 105–160.

[3] Adem, A. and R. J. Milgram, R. J. *Cohomology of Finite Groups* (Springer-Verlag Grundlehren 309), Springer-Verlag, Berlin, 1994.

[4] Benson, D. and Carlson, J. *Projective Resolutions and Poincaré Duality Complexes*, Transactions of the A.M.S. **342** (1994), pp. 447–488.

[5] Browder, W. and Pakianathan, J. *Cohomology of Uniformly Powerful p-groups*, Trans. AMS **352** (2000), pp. 2659–2688.

[6] Cenkl, B. and Porter, R. *Cohomology of Nilmanifolds* Algebraic topology–rational homotopy, Springer–Verlag LNM **1318** (1988).

[7] Duflot, J. *Depth and Equivariant Cohomology*, Comm. Math. Helv. **56** (1981), pp. 627–637.

[8] Evens, L. *Cohomology of Groups*, Oxford University Press, Oxford, 1991.

[9] Gugenheim, V. and May, P. *On the Theory and Applications of Torsion Products*, AMS Memoirs **142** (1974).

[10] Jozefiak, T. and Weyman, J. *Representation–theoretic interpretation of a formula of D.E. Littlewood*, Math Proc. Cambridge Phil. Soc. **103** (1988), pp. 193–196.

[11] Lambe, L. and Priddy, S. *Cohomology of nilmanifolds and torsion-free nilpotent groups*, Tr. AMS **273** (1982), pp. 39–55.

[12] Serre, J.-P. *Algèbre Locale—Multiplicités*, Springer Lecture Notes in Mathematics, **11**, Springer-Verlag, Berlin, 1965.

[13] Sigg, S. *The Laplacian and the Homology of Free Two-step Nilpotent Lie Algebras*, J. Algebra **185** (1996), pp. 144–161.

[14] Weigel, T. *P–central Groups and Poincaré Duality*, Trans. AMS (to appear).

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